Effective Boundary Conditions for Continuum Method of Investigation of Rarefied Gas Flow over Blunt Body

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Abstract. Super- and hypersonic rarefied gas flow over blunt bodies is investigated by using asymptotically correct viscous shock layer (VSL) model with effective boundary conditions and thin viscous shock layer model. Correct shock and wall conditions for VSL are proposed with taking into account terms due to the curvature which are significant at low Reynolds number. These conditions improve original Davis's VSL model [1]. Numerical calculation of Krook equation [2] is carried out to verify continuum results. Continuum numerical and asymptotic solutions are compared with kinetic solution, free-molecule flow solution and with DSMC solutions [3, 4, 5] over a wide range of free-stream Knudsen number Kn_∞. It is shown that taking into account terms with shock and surface curvatures have a pronounced effect on skin friction and heat-transfer in transitional flow regime. Using the asymptotically correct VSL model with effective boundary conditions significantly extends the range of its applicability to higher Kn_∞ numbers.

Keywords: hypersonic rarefied flow, transitional regime, slip effects, continuum model, heat transfer.
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INTRODUCTION

The problem of aerothermodynamics in hypersonic rarefied flow is important for space vehicles reentry at high altitudes. In transitional to free-molecule flow regime, corresponding to high Knudsen number Kn, or low Reynolds number Re, Direct Simulation Monte Carlo methods (DSMC) are usually applied or kinetic and various hybrid methods are used which join solution of kinetic equations, or DSMC solution, with solution of continuum equations. Restriction on application of continuum-flow models in a rarefied gas flow does not exclude using the continuum approach for prediction of some hypersonic flow parameters. In [6] it was shown that two continuum-flow models can be used for pressure, heat-transfer and skin-friction coefficients prediction in transitional flow regime: viscous shock layer (VSL) and thin viscous shock layer (TVSL). These models were proposed – VSL in [1], TVSL in [7] – for moderately high Re number. In [8] it was shown by asymptotic analysis of the Navier – Stokes (NS) equations that VSL and TVSL equations are valid also for low Re number. For high Re number TVSL give incorrect pressure prediction far from a stagnation point, especially for such bodies as spheres, so the domain of validity of this model is limited from above as respects Re number. At the same time TVSL give correct pressure, heat-transfer and skin-friction coefficients at low Re number and correct free-molecule limits for these coefficients (at unit accommodation coefficient) as Re number tends to zero, while the domain of validity of VSL is limited from below.

In previous work [6] the original VSL model [1] with taking into account slip velocity and temperature jump on a wall was used in transitional flow regime and it was shown, that it gives good heat transfer prediction for cold surface in hypersonic flow up to free-stream Kn number $Kn_\infty \sim 15$. But at higher Kn_∞ number this model gave excess over free-molecule limit and further unlimited growth of heat-transfer coefficient. In addition solution for skin friction was not enough accurate, especially for sphere. The purpose of the present work is to derive the asymptotically correct boundary conditions at the shock and at the wall for VSL at low Re number in order to improve original VSL model [1], correct skin friction prediction and extend a range of applicability of VSL. To facilitate solving the considered problem numerical calculation of Krook equation [2] is carried out. To verify continuum results and estimate the effect of various terms in boundary conditions on solution, continuum numerical and asymptotic solutions are compared with kinetic solution, free-molecule flow solution and with available in literature solutions, obtained by the DSMC method over a wide range of Kn_∞.

ASYMPTOTICALLY CORRECT EQUATIONS AND BOUNDARY CONDITIONS AT LOW REYNOLDS NUMBER

Two continuum models, VSL and TVSL, are used for study of axisymmetric and plane hypersonic rarefied gas flows over blunt bodies. Asymptotic analysis of the NS equations in hypersonic shock layer [8] showed that VSL and TVSL equations are valid not only at high, but also at low Re number under the assumption that introduced parameter χ is small (this parameter is of the order of shock layer thickness). TVSL equations are derived from the NS equations at low Re number by neglecting terms $O(\chi)$ except tangential pressure gradient term $O(\chi)$, which is taken into account in order to extend the range of TVSL validity to high Re numbers. TVSL equations and shock conditions at low Re number are the same as in Cheng's model [7] proposed for moderately high Re number (in this model tangential pressure gradient is also of the higher order). Slip velocity and temperature jump are $O(\chi)$ [6] and so they should be neglected. Thus the asymptotically correct TVSL model at low Re number is without tangential pressure gradient term and wall slip effects. This model gives correct free-molecule limits for heat-transfer, skin friction and pressure coefficients as $\mathrm{Re} \to 0$ which is verified by numerical and asymptotic solutions.

VSL equations are derived from the NS equations at low Re number by neglecting terms $O(\chi^2)$ and taking into account terms $O(\chi)$ and they are the same as in original Davis's VSL model [1]. But shock and wall boundary conditions at low Re number differ from the conditions [1]. This work is mainly focused on asymptotically correct shock and wall conditions for VSL equations.

The modified Rankine-Hugoniot relations at the shock for VSL at low Re number are derived from the general relations at the shock [9] by neglecting terms $O(\chi^2)$ and they are of the form

$$
\begin{aligned}
&v_s = u_s \,\mathrm{tg}\,\beta_s - \frac{1}{\rho_s}\frac{\sin\beta}{\cos\beta_s}, \quad \beta_s = \beta - \alpha \\
&u_S = \cos\beta\cos\beta_S + \frac{1}{\rho_S}\sin\beta\sin\beta_S - \frac{\mu_S\cos^3\beta_s}{\mathrm{Re}_\infty\sin\beta}\left(\frac{\partial u}{\partial y} - \frac{u}{RH_1}\right)_S \\
&T_S + \frac{1}{2}\left(u_S^2 + v_S^2\right) = \frac{1}{2} + \frac{1}{(\gamma-1)\mathrm{M}_\infty^2} - \frac{\mu_S\cos\beta_s}{\mathrm{Re}_\infty\sin\beta}\left[\frac{1}{\mathrm{Pr}}\frac{\partial T}{\partial y} + u\left(\frac{\partial u}{\partial y} - \frac{u}{RH_1}\right)\right]_S \\
&p_S = \left(1 - \frac{1}{\rho_S}\right)\sin^2\beta + \frac{1}{\gamma\mathrm{M}_\infty^2}
\end{aligned}
\tag{1}
$$

Here $\beta(x)$ and $\beta_s(x)$ are angles of inclination of a shock $y = y_s(x)$ to axis of symmetry and to axis x, $\mathrm{d}y_s/\mathrm{d}x = H_{1s}\tan\beta_s$, $H_1 = 1+y/R$ – Lame coefficient, α – an angle between a tangent to surface contour and free-stream velocity $\boldsymbol{V}_\infty$, $V_\infty u$ and $V_\infty v$ – tangential and normal velocity components, $\rho\rho_\infty$ – density, $\mu\mu(T_\infty)$ – viscosity, TV_∞^2/c_p – temperature, $\rho_\infty V_\infty^2 p$ - pressure; RR_0 – radius of a curvature, γ - specific heats ratio, Pr - Prandtl number, M_∞ – free-stream Mach number, R_0 – nose radius; x and y – tangential and normal non-dimensionalized by R_0 coordinates.

Conditions (1) improve Davis's conditions [1] in which as in many following studies factors $\cos^3\alpha$ and $\cos\alpha$ in the momentum and energy equations are missed by neglecting difference between normals to a shock and to a body. These factors should be present at low as well as at high Re number. Conditions (1) also take into account terms $u/(RH_1)$ and $u^2/(RH_1)$ due to curvature which are of the same order of magnitude as other terms at low Re number being of the higher order at high Re number. These terms have an effect on flow parameters in transitional regime.

Effective wall boundary conditions for slip velocity and temperature jump with taking into account terms due to the surface curvature are proposed, with using relations for rectilinear surface [10]

$$
\begin{aligned}
&u = \frac{2-\theta}{\theta}\sqrt{\frac{\pi}{2}}\frac{\mu}{\varepsilon^{1/2}\mathrm{Re}_\infty T^{1/2}\rho}\left(\frac{\partial u}{\partial y} - \frac{u}{R}\right) \\
&T = T_w + \frac{2-\alpha'}{\alpha'}\frac{2\gamma}{(\gamma+1)}\sqrt{\frac{\pi}{2}}\frac{\mu}{\varepsilon^{1/2}\mathrm{Re}_\infty T^{1/2}\rho}\left(\left(\frac{T_w}{T}\right)^\alpha\frac{1}{\mathrm{Pr}}\frac{\partial T}{\partial y} + u\left(\frac{\partial u}{\partial y} - \frac{u}{R}\right)\right)
\end{aligned}
\tag{2}
$$

Here, $T_w V_\infty^2/c_p$ – wall temperature, $\varepsilon = (\gamma-1)/(2\gamma)$. These wall conditions as well as shock conditions are different from conditions [1], in which there are no terms u/R and $u(\partial u/\partial y - u/R)$ and the factor $(T_w/T)^\alpha$ before temperature gradient. The factor T_w/T [10] is a term of the higher order on the assumption of small slip effects when this factor is assumed to be equal to 1, but at high Kn number this factor should be taken into account because its effect on some flow parameters is significant. We have introduced power parameter α to be in accord with various conditions in transitional flow regime ($\alpha = 0$ corresponds to factor $T_w/T = 1$).

CONTINUUM CALCULATION METHODS

Numerical and asymptotic methods are used for solving the problem of rarefied gas flow over blunt body by using continuum models. Numerical VSL and TVSL solutions are obtained by using the low-iterative high-resolution fully coupled implicit space-marching procedure. To take into account upstream influence, the accelerated method of global iterations on an elliptical component of pressure gradient is elaborated. An effective splitting of a tangential pressure gradient into hyperbolic and elliptic components is employed.

The analytical solution for heat-transfer, skin-friction and pressure coefficients obtained by asymptotic method is used at very low Re number:

$$C_H = \sin\alpha\left[1 - \frac{1+\omega}{3(2-\omega)}\mathrm{Pr}\,\tau\right] + O(\tau^2)$$

$$C_f = 2\sin\alpha\cos\alpha\left[1 - \frac{1}{3}\left(\frac{1+\omega}{2-\omega} + \frac{\sin\alpha}{R\beta^*}\right)\tau\right] + O(\tau^2), \quad p_w = \sin^2\alpha - \frac{\sin\alpha\cos^2\alpha}{3R\beta^*}\tau \qquad (3)$$

$$\tau = \mathrm{Pr}^{\frac{1-\omega}{1+\omega}}(\varepsilon\,\mathrm{Re}/\beta^*)^{1/(1+\omega)}, \quad \beta^* = \frac{1}{2}\left(\frac{\sin\alpha}{R} + \nu\frac{\sin\alpha\cos\alpha}{r_w}\right)$$

Here α is an angle between a tangent to the surface contour and the axis of symmetry, RR_0 – radius of curvature, $r_w R_0$ – a distance from a surface to an axis of symmetry, $\nu = 1, 0$ corresponds to axisymmetric and plane flow. $\mathrm{Re}=V_\infty\rho_\infty R_0/\mu(T_0)$, T_0 – freestream stagnation temperature, $\mu \sim T^\omega$ is assumed. Solution (3) gives correct free-molecule limits at unit accommodation coefficient as $\mathrm{Re} \to 0$.

SOLUTION OF KROOK EQUATION

The numerical calculation of Krook kinetic equation [2] is carried out to solve the problem of hypersonic rarefied gas flow near a circular cylinder. Because of its simplicity, the kinetic model BGK [2], known as Krook equation, is widely applied. Unsteady Krook equation is solved by using implicit numerical Euler scheme of the first order approximation on space and time coordinates. The model of hard spheres is used for molecule interaction. Diffusion molecule scattering with full thermal accommodation to surface temperature is accepted as condition on the surface of a cylinder. Freestream conditions are defined by number density n_∞, temperature T_∞ and velocity U_∞.

A transition from low Kn number to large is of special interest. Knudsen layer thickness is of order of free path. In the continuum flow regime Knudsen layer thickness is less than a cell of computational grid. The values of macroscopic parameters nearby a body surface correspond to tens of free paths. As Kn number increase, a cell of computational grid becomes comparable with Knudsen layer thickness. It enables to obtain more detailed information about macroscopic parameters close to a surface.

COMPARISON AND DISCUSSION OF RESULTS

Numerical and analytical VSL and TVSL solutions are compared with solutions obtained by the DSMC method, with free-molecule flow solution and with kinetic solution. To estimate the effect of various terms in shock and wall conditions on solution different VSL solutions have been obtained: original VSL model [1] and extended VSL with boundary conditions (1), (2) with various values of parameter α in slip wall conditions ($\alpha = 0, 0.5, 1$).

When comparing with DSMC results [3, 4, 5], freestream conditions correspond to Space Shuttle Orbiter during reentry at altitudes H = 90-150 km (V_∞ = 7.5 km/s). Heat transfer coefficient C_H and skin friction coefficient C_f prediction on a sphere with R_0 = 0.0254 m (T_w/T_0 = 0.07) at altitudes H = 90, 100, 110 km (Kn_∞ = 0.9, 5.4, 30.4) are compared with DSMC results [3] in Fig. 1. Comparison with DSMC solution [4] for C_H and C_f on 50° hyperboloid with R_0 = 1.143 m (T_w = 0.02-0.05) modeling the windward centerline of the Shuttle at 40° angle of incidence at H = 140 and 110 km (Kn_∞ = 15.4 and 0.67) is shown in Fig. 2 ($\mu \sim T^\omega$ is assumed, ω = 0.73). Comparison with DSMC results [5] for temperature jump prediction at a stagnation point of 42.5° hyperboloid with R_0 = 1.362 m versus Kn_∞ is demonstrated in Fig. 3. Calculations correspond to flight altitudes H = 110, 122.5, 130, 150 km.

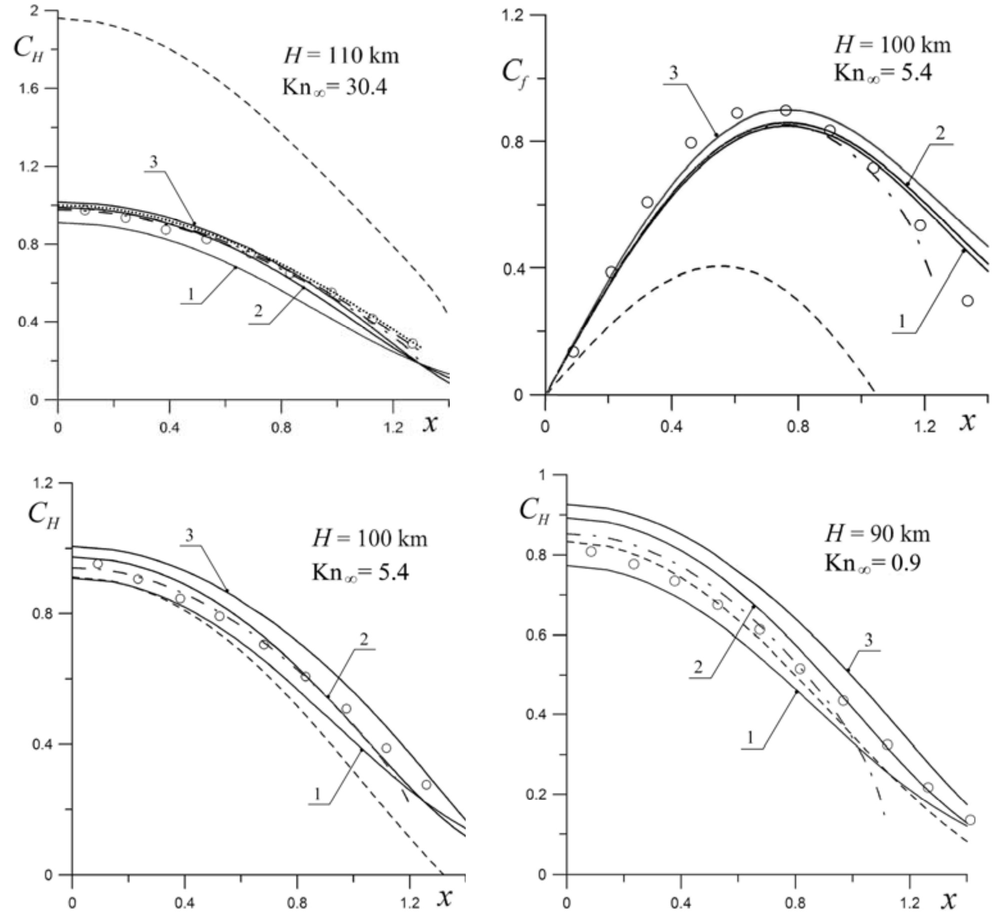
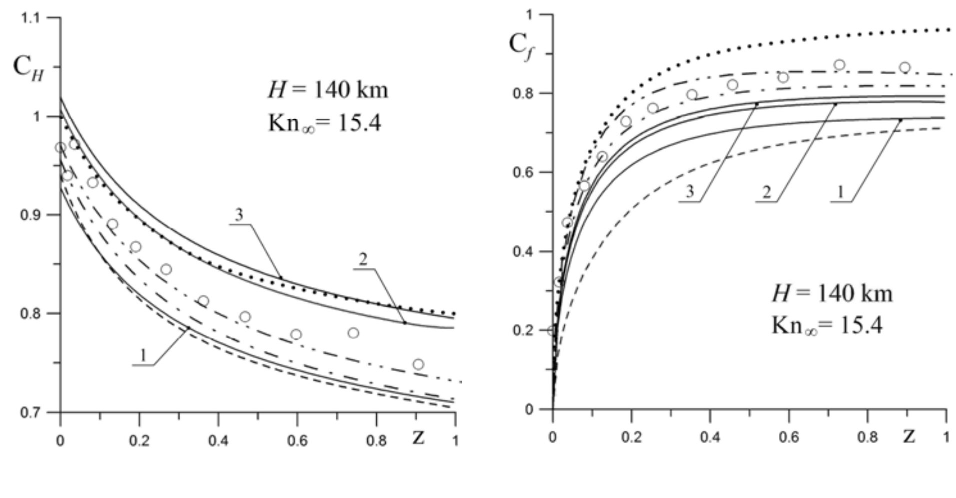


FIGURE 1. C_H and C_f distributions on the sphere. Dash line – original VSL; solid lines 1, 2, 3 – VSL with conditions (1)-(2), α = 0, 0.5, 1; dash-and-dot line – TVSL; double dash-and-dot line – formulae (3); dot line – free-molecule flow; circles – DSMC [3].



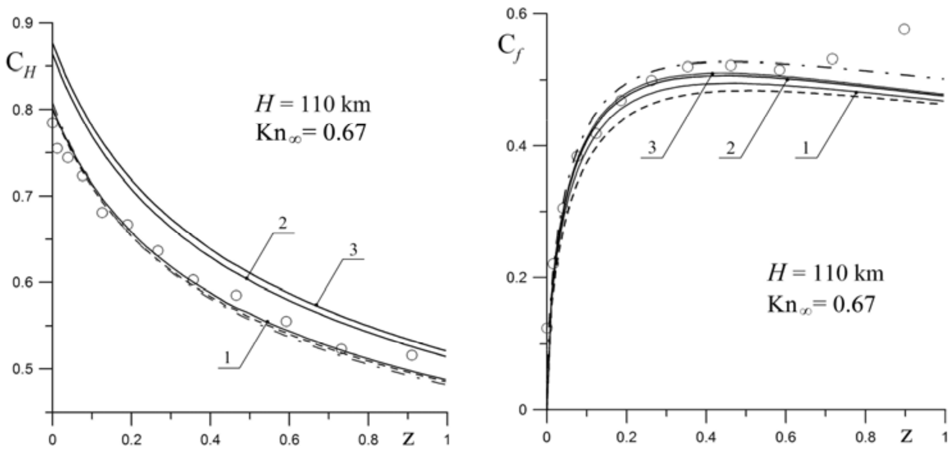


FIGURE 2. C_H and C_f distributions on 50° hyperboloid, z – distance from a stagnation point along axis of symmetry in meters. Dash line – original VSL; solid lines 1, 2, 3 – VSL with conditions (1)-(2), α = 0, 0.5, 1; dash-and-dot line – TVSL; double dash-and-dot line – formulae (3); dot line – free-molecule flow; circles – DSMC [4].

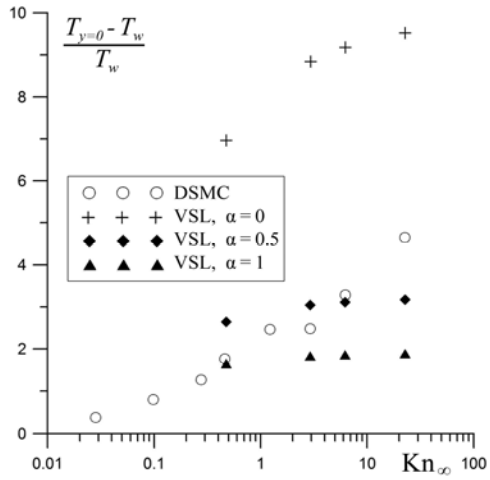


FIGURE 3. Temperature jump at a stagnation point of 42.5° hyperboloid versus Kn_∞. Circles – DSMC [5].

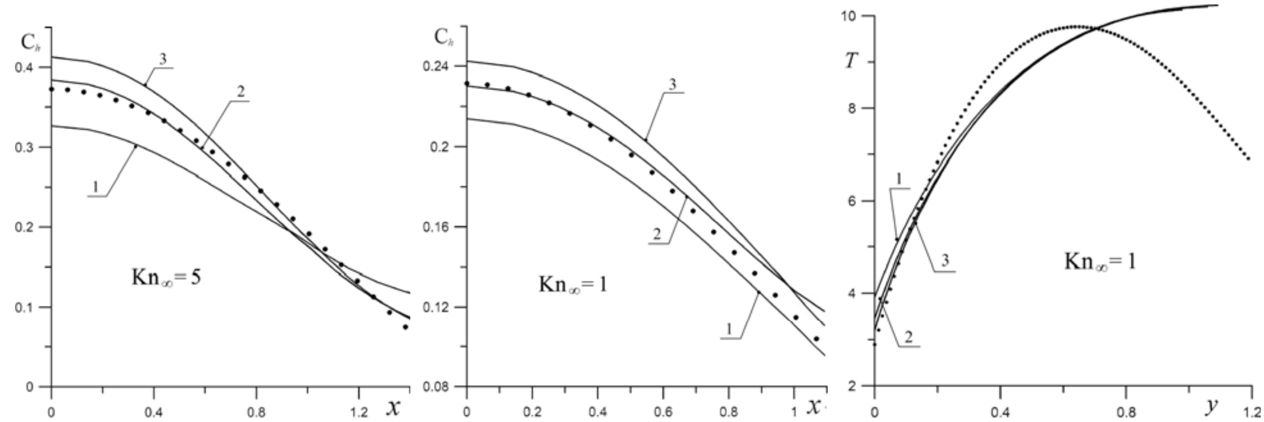


FIGURE 4. C_h distributions on circular cylinder at Kn_∞ = 1, 5 and T distribution along a stagnation line at Kn_∞ = 1. Lines 1, 2, 3 – VSL with conditions (1)-(2), α = 0, 0.5, 1; dots – kinetic solution

Comparison with solution of Krook equation for C_h on circular cylinder and temperature T (non-dimensionalized by freestream temperature T_∞) distribution along the normal to a body at a stagnation point is represented in Fig. 4. Calculation carried out at Kn_∞ = 1 and 5, M_∞ = 5.5 and wall temperature equal to T_∞.

Comparisons represented in Figs. 1-2 demonstrate that taking into account terms due to curvature in shock and wall conditions greatly improve skin friction prediction in transitional flow regime, especially for such body as sphere. The effect of curvature terms on heat transfer prediction is not so large at moderately high Kn_∞ number vanishing at $Kn_\infty \sim 1$, but at very high Kn_∞ number the significance of curvature terms in boundary conditions is very great. Original VSL model gives correct C_H prediction for cold wall up to $Kn_\infty \sim 15$, but with the increase of Kn_∞ this model gives C_H value that increases exceeding free-molecule limit as shown in Fig. 1 for Kn_∞ = 30.4, so the model is no longer valid. However using effective boundary conditions with curvature terms make it possible to predict heat transfer rather correctly up to Kn_∞ = 30. The restriction on using VSL model at high Kn_∞ number is connected rather with computational difficulties than with wrong prediction. At very high Kn_∞ number the most accurate C_H and C_f values are given by the simpler TVSL model (giving correct free-molecule limits for these coefficients as $Kn_\infty \to \infty$), and the analytical solution (3) is very accurate (Figs. 1, 2).

Comparison with kinetic solution in Fig. 4 shows that VSL at α = 0.5 gives good heat transfer prediction at $Kn_\infty \sim$ 1-5 and satisfactory temperature distribution along the stagnation line near the body at $Kn_\infty \sim 1$, although in this case of plane flow and not high M_∞ number the shock layer is rather thick. Figure 3 shows that temperature jump prediction strongly depend on parameter α and at various Kn_∞ the best prediction is obtained at various α. Although in the case of the cold wall ($T_w/T_0 < 0.1$) the heat transfer coefficient depends weakly on wall temperature and the accuracy of heat transfer prediction is connected rather with using asymptotically correct model than with accuracy of temperature jump prediction, the correct wall conditions are very important. Given all the comparisons carried out, it seems worth to obtain more precise wall conditions for VSL equations, possibly with dependence of parameter α on Kn_∞ number and with taking into account additional terms with temperature gradient in velocity slip equation and ones with velocity gradient in temperature jump equation.

CONCLUSION

The asymptotically correct shock and wall boundary conditions for VSL equations with taking into account terms due to shock and surface curvatures are proposed. They improve the original VSL model [1], considerably correcting the heat-transfer and especially skin friction prediction and significantly extending the range of applicability of VSL to higher Kn_∞ numbers. Thus for heat transfer and skin friction prediction in transitional flow regime the extended VSL model with conditions (1), (2) can be used while at very high Kn_∞ numbers the TVSL model or analytical solution (3) can be used. For a more accurate prediction of some flow parameters, in particular temperature and velocity jump, further refinement of wall conditions would be useful.

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